

UT-09-08  
March 2009

# Charges and homologies in $\text{AdS}_4/\text{CFT}_3$

Yosuke Imamura<sup>\*</sup>

*Department of Physics, University of Tokyo,  
Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan*

## Abstract

Electric and magnetic charges of a certain class of operators in  $\mathcal{N} = 2$  large  $N$  quiver Chern-Simons theories are investigated. We consider only non-chiral theories, in which every bi-fundamental field appears with its conjugate representation. By interpreting operators in a Chern-Simons theory as wrapped M-branes in the dual geometry  $\text{AdS}_4 \times X_7$ , we partly determine the homologies of  $X_7$ .

---

<sup>\*</sup>E-mail: [imamura@hep-th.phys.s.u-tokyo.ac.jp](mailto:imamura@hep-th.phys.s.u-tokyo.ac.jp)

# 1 Introduction

$AdS_4/CFT_3$  correspondence claims the equivalence between a three-dimensional conformal field theory (CFT) and M-theory in the background

$$AdS_4 \times X_7, \tag{1}$$

with an appropriately chosen Einstein 7-manifold  $X_7$ , which may in general contain singularities. In the brane picture, the CFT is the theory realized on multiple M2-branes in the cone over  $X_7$ . Until quite recent, low-energy effective field theories realized on multiple M2-branes were not known. The first example of interacting CFTs describing multiple M2-branes were proposed by Bagger and Lambert [1, 2, 3] and Gustavson [4, 5]. The model, which is now called BLG model, is a Chern-Simons theory with  $\mathcal{N} = 8$  supersymmetry, and is important also as the first example of Chern-Simons theories with  $\mathcal{N} \geq 4$  supersymmetry. Although BLG model works only for two M2-branes [6, 7], it triggered the following construction and classification of various supersymmetric quiver Chern-Simons theories [8, 9, 10, 11, 12, 13, 14, 15].  $\mathcal{N} \leq 3$  quiver Chern-Simons theories, whose actions have been known for long, are also investigated from new perspective as theories for multiple M2-branes in various backgrounds [16, 17, 18, 19, 20, 21, 22, 23, 24].

Almost all theories appearing in the recent literature as theories on M2-branes are quiver Chern-Simons theories.<sup>1</sup> The purpose of this paper is to investigate the relation between charges in such quiver Chern-Simons theories and wrapped M-branes in the corresponding internal space  $X_7$ .

In the case of  $AdS_5/CFT_4$  duality, the relation between wrapped branes and operators are first proposed in [27] for non-dynamical (external) baryonic operators in the maximally supersymmetric Yang-Mills theories, and was extended to dynamical baryonic operators in theories with less supersymmetries in [28, 29]. In the latter, baryonic operators are identified with D3-branes wrapped on three-cycles in internal five-dimensional spaces. This is further extended in [30, 31, 32] to a large class of quiver gauge theories described by brane tilings [33, 34, 35]. In such a class of gauge theories, the spectrum of baryonic operators is known to be consistent with the homology of three-cycles, on which D3-branes can wrap.

The relation between fractional branes, D5-branes wrapped on two-cycles, and ranks of  $SU(N)$  gauge groups is also well understood [36, 37] in the case of Klebanov-Witten theory [38]. See also [39, 40] for fractional branes in more general theories described by brane tilings.

In this paper, we discuss similar relations for  $AdS_4/CFT_3$  from the field theory side. We investigate electric and magnetic charges of operators, and relate them

---

<sup>1</sup>Recently,  $\mathcal{N} = 3$  Chern-Simons theories with fundamental flavors describing multiple M2-branes in hyper-Kähler cone backgrounds were proposed in [25, 26]. We do not discuss such theories in this paper.

to wrapping numbers of M5- and M2-branes. We determine one-, two-, and five-cycle homologies of  $X_7$  by the analysis of charges on the field theory side. We also discuss the relation between ranks of gauge groups and the four-form flux in the internal space.

We emphasize that operators considered in this paper are not always gauge invariant. In general, they are charged under the  $U(1)$  part of  $U(N) \sim SU(N) \times U(1)$  gauge groups. We do not impose gauge invariance with respect to all these  $U(1)$  subgroups. This is because, as is pointed out in [41], it is impossible to identify all wrapped M-branes which are particles in  $AdS_4$  with gauge invariant operators. We require operators to be singlet with respect to  $SU(N)$  subgroups in the gauge group. We refer to such  $SU(N)$  invariant operators as “colorless operators.”

Colorless operators are constructed by contracting all  $SU(N)$  color indices of constituent objects. For simplicity, we do not consider the full symmetry structure of  $SU(N)$  indices. For example, when an operator has two color indices, we do not take care about whether the indices are symmetric or anti-symmetric. We only take account of the “index number”  $z$  defined by

$$z = \# \text{ of upper } SU(N) \text{ indices} - \# \text{ of lower } SU(N) \text{ indices.} \quad (2)$$

We determine whether a combination of operators can be colorless by only checking whether the total index number vanishes. Of course, this simple prescription cannot capture the detailed spectrum of operators, and more careful analysis is necessary when we want to determine degeneracy of operators and so on. We leave this problem for future work.

As we mentioned above, colorless operators are in general charged under the diagonal  $U(1)$ ’s of  $U(N)$ ’s in the gauge group. We call this baryonic symmetry, and refer to operators rotated by this symmetry as baryonic operators. Such operators are constructed with  $SU(N)$  epsilon tensors, and expected to have conformal dimension of order  $N$ . It is known that the mass of M5-branes wrapped on five cycles reproduce this scaling of the conformal dimension, and by this reason, we identify baryonic operators with wrapped M5-branes. Once we accept this correspondence, it is natural to identify monopole operators, operators magnetically charged with respect to the baryonic symmetry, with M2-branes wrapped on two-cycles, which are mutually non-local to the wrapped M5-branes.

In general, quantum corrections shift the baryonic charges of monopole operators [42, 43], and they make the charge spectrum of such operators complicated. Unfortunately, we have not succeeded in interpreting such a complicated spectrum in terms of M-branes. By this reason, in this paper, we discuss only the non-chiral  $\mathcal{N} = 2$  quiver Chern-Simons theories, in which a bi-fundamental chiral multiplet in  $(N_a, \bar{N}_b)$  and one in  $(\bar{N}_a, N_b)$  appear in the pairwise way and the corrections to baryonic charges vanish. Examples of such non-chiral theories describing M2-branes are  $\mathcal{N} = 3$  Chern-Simons theories studied in [17].

This paper is organized as follows. In §2 we define the colorless sector for a single  $U(N)$  gauge group. In §3 we generalize this into quiver Chern-Simons theories, and declare the class of operators we discuss. In §4 we relate M2-branes wrapped on two-cycles to non-baryonic monopole operators. In §5, we discuss the correspondence between baryonic operators and wrapped M5-branes. In §6 we study how flux strings are realized as wrapped M-branes. (By flux strings we mean stringy objects in  $AdS_4$ . Since we consider CFT, confining strings do not exist on the field theory side.) A relation among ranks of gauge groups, baryonic charges, and the charge of flux strings attached on baryonic operators are also studied. In §7 we present some examples. The last section is devoted to discussions.

## 2 $U(N)$ gauge group

In this preliminary section, we consider a gauge theory with a single  $U(N)$  gauge group. We denote this gauge group by  $G$ , and the gauge field by  $A$ . We assume that  $U(N)$  is the effective gauge group. If there were no matter fields the gauge group would be effectively  $SU(N)/\mathbb{Z}_N$  because the diagonal  $U(1)$  would not couple to any fields. We do not consider such a case and assume the existence of fields coupled by the full  $U(N)$  gauge group.

We can use Young diagrams to specify  $U(N)$  representations. Let  $w_i$  be the number of boxes in the  $i$ -th row in a Young diagram. These numbers form the highest weight vector for the representation. It is an element of the  $U(N)$  weight lattice  $\mathcal{W}$ .

$$\vec{w} = (w_1, w_2, \dots, w_N) \in \mathcal{W}. \quad (3)$$

When we use a weight vector to specify Young diagram, the components are ordered in the descending order;  $w_1 \geq w_2 \geq \dots \geq w_N$ . Because we consider not  $SU(N)$  but  $U(N)$ , columns filled up with  $N$  boxes have meaning and we cannot neglect them. In other words, we should distinguish between  $(w_1, \dots, w_N)$  and  $(w_1 + 1, \dots, w_N + 1)$ . Note that negative  $w_i$  are not prohibited. Although the  $SU(N)$  representation and the  $U(1)$  charge are independent for general  $U(N)$  representations, we consider only the case in which both the  $U(1)$  charge and the  $SU(N)$  representation are specified by the same Young diagram.

We define monopoles in  $U(N)$  gauge theory following Goddard, Nuyts, and Olive [44]. Namely, we consider Dirac monopoles for the Cartan subgroup  $H = U(1)^N \subset G$ . Such monopoles are characterized by the Dirac strings attached on them, and their magnetic charges are defined by integrating the gauge potential around the Dirac string as

$$m_i = \frac{1}{2\pi} \oint A_i, \quad i = 1, \dots, N, \quad (4)$$

where the index  $i$  labels the  $U(1)$ 's in  $H$ , and  $A_i$  is  $i$ -th diagonal component in the  $U(N)$  gauge field  $A$ . Monopoles defined in this way are called Goddard-Nuyts-Olive (GNO) monopoles. We can regard the set of magnetic charges  $m_i$  as a vector in the root lattice  $\mathcal{R}$ :

$$\vec{m} = (m_1, m_2, \dots, m_N) \in \mathcal{R}. \quad (5)$$

The weight lattice  $\mathcal{W}$  and the root lattice  $\mathcal{R}$  are dual to each other, and for vectors  $\vec{w} \in \mathcal{W}$  and  $\vec{m} \in \mathcal{R}$  the inner product

$$\vec{w} \cdot \vec{m} \equiv \sum_{i=1}^N w_i m_i \quad (6)$$

is defined.

We mainly focus on “the colorless sector” as is mentioned in Introduction. This means that we focus on the electric and magnetic charges with respect to the diagonal  $U(1)$  subgroup of  $U(N)$ . When we discuss electric charge in the colorless sector, we consider only  $SU(N)$  singlet operators.  $SU(N)$  non-singlet representations are excluded from the consideration. This constrains weight vectors by “the colorless condition”

$$\vec{w} \cdot \vec{\alpha}_a = 0, \quad a = 1, \dots, N-1, \quad (7)$$

where  $\vec{\alpha}_a \in \mathcal{R}$  are the  $SU(N)$  root vectors defined by

$$\begin{aligned} \vec{\alpha}_1 &= (1, -1, 0, \dots, 0), \\ \vec{\alpha}_2 &= (0, 1, -1, 0, \dots, 0), \\ &\vdots \\ \vec{\alpha}_{N-1} &= (0, \dots, 0, 1, -1). \end{aligned} \quad (8)$$

The colorless condition (7) means

$$w_1 = w_2 = \dots = w_N =: b, \quad (9)$$

and the Young diagram of a colorless representation is an  $N \times b$  rectangle.

If we consider only colorless operators, the effective gauge group becomes

$$G_B = U(N)/SU(N) = U(1)/\mathbb{Z}_N, \quad (10)$$

where  $U(1)$  in the last expression means the diagonal  $U(1)$  subgroup of  $U(N)$  and  $\mathbb{Z}_N$  is the center of  $SU(N)$ . We define the  $G_B$  gauge field by

$$B = \text{tr } A. \quad (11)$$

This couples to the fundamental representation by charge  $1/N$ .

Contrary to the electric charges, we do not impose any restriction to the magnetic charge  $\vec{m}$ . Instead, we simply neglect the magnetic charges other than the  $G_B$  charge. This is realized by introducing the following equivalence relation:

$$\vec{m} \sim \vec{m} + \sum_{a=1}^{N-1} c_a \vec{\alpha}_a, \quad c_a \in \mathbb{Z}. \quad (12)$$

This identification removes  $N-1$  components of the magnetic charges, and leaves information of the  $G_B$  magnetic charge only.

In general, when we consider a pairing of two linear spaces with inner product between them, an equivalence relation in one space always arises with a constraint in the other space for the consistency with the inner product. In the case of charge lattices we discuss here, the inner product  $\vec{w} \cdot \vec{m}$  is well-defined in the colorless sector because it does not depend on the choice of an element from an equivalence class defined by (12) thanks to the restriction (7).

When we consider the colorless sector, we can use a single integer to represent each of electric and magnetic charges. For the electric charge, we use the common value  $b$  in (9), while an equivalence class of the  $G_B$  magnetic charge defined by (12) is specified by

$$m = \sum_{i=1}^N m_i. \quad (13)$$

The inner product of electric and magnetic charge vectors is equal to the product of these integers.

$$\vec{w} \cdot \vec{m} = bm. \quad (14)$$

For concreteness, let us consider  $U(N)$  gauge theory with chiral multiplets  $Q^\alpha$  and  $\tilde{Q}_\alpha$  in the fundamental and anti-fundamental representation, respectively. We also assume the existence of the Chern-Simons term

$$S_{\text{CS}} = \frac{k}{4\pi} \int \text{tr} \left( AdA + \frac{2}{3} A^3 \right). \quad (15)$$

Colorless operators in this Chern-Simons theory are constructed by combining the following objects:

- The component fields in  $Q^\alpha$  and  $\tilde{Q}_\alpha$ .
- $SU(N)$  invariant anti-symmetric tensors  $\epsilon_{\alpha_1 \dots \alpha_N}$  and  $\epsilon^{\alpha_1 \dots \alpha_N}$ .
- Monopole operators.

If the Chern-Simons level is  $k$ , a monopole operator  $\mathbf{m}[\vec{m}]$  with magnetic charge  $\vec{m}$  belongs to  $SU(N)$  representation specified by the weight vector  $\vec{w} = k\vec{m}$ . In general, the  $G_B$  charge of the operator receives quantum corrections.

As we mentioned in Introduction, we consider only non-chiral theories in which such corrections vanish. Then the  $G_B$  charge is given by  $b = km/N$ , and we can regard  $\vec{w}$  as a  $U(N)$  weight vector.

The above mentioned monopole operators are elementary ones before combined with matter fields to form colorless operators. We use the character  $\mathbf{m}$  to denote such “bare” monopole operators. The index number and the electric and the magnetic charges of these objects are shown in Table 1. We construct col-

Table 1: The index number  $z$ , the electric  $G_B$  charge  $b$ , and the magnetic  $G_B$  charge  $m$  of elementary objects are shown.

	$z$	$b$	$m$
$Q^\alpha$	1	$1/N$	0
$\tilde{Q}_\alpha$	-1	$-1/N$	0
$\epsilon_{\alpha_1 \dots \alpha_N}$	$-N$	0	0
$\epsilon^{\alpha_1 \dots \alpha_N}$	$N$	0	0
$\mathbf{m}[\vec{m}]$	$km$	$km/N$	$m$

orless operators by combining these objects so that the index number  $z$  cancels. For such operators, the electric charge  $b$  is always an integer and is the same as the number of the epsilon tensor. (We mean by “the number of the epsilon tensor” the number of  $\epsilon_{\alpha_1 \dots \alpha_N}$  subtracted by that of  $\epsilon^{\alpha_1 \dots \alpha_N}$ .) Namely, the charge  $b$  counts the number of  $SU(N)$  “baryons”. This is the reason why we call  $G_B$  the baryonic symmetry.

### 3 Quiver Chern-Simons theories

Let us extend the arguments in the last section to quiver gauge theories. We consider a quiver Chern-Simons theory described by a connected quiver diagram with  $n$  vertices. The gauge group is given by

$$G = \prod_{a=1}^n U(N_a), \quad (16)$$

and the action includes the Chern-Simons terms

$$S_{\text{CS}} = \sum_{a=1}^n \frac{k_a}{4\pi} \int \text{tr} \left( A_a dA_a + \frac{2}{3} A_a^3 \right). \quad (17)$$

We define the “color part” of the gauge group by

$$G_{\text{SU}} = \prod_{a=1}^n SU(N_a) \subset G. \quad (18)$$

Note that we do not remove the diagonal  $U(1)$  subgroup of  $G$  which does not act on any matter fields in the gauge theory. This is because we implicitly assume that the theory is embedded in string or M-theory. In such a case we can introduce an external source belonging to the fundamental representation in a  $U(N_a)$  gauge group, to which the diagonal  $U(1)$  subgroup couples. We later impose a certain condition (eq. (24)) to exclude such representation from the physical spectrum.

We consider the colorless sector of this quiver gauge theory. The baryonic symmetry  $G_B$  is defined as the effective group acting on colorless operators:

$$G_B = G/G_{\text{SU}} = \prod_{a=1}^n (U(1)'_a / \mathbb{Z}_{N_a}) = \prod_{a=1}^n U(1)_a, \quad (19)$$

where  $U(1)'_a$  is the diagonal subgroup of  $U(N_a)$  and  $U(1)_a$  is its quotient by  $\mathbb{Z}_{N_a}$ , the center of  $SU(N_a)$ . Let  $B_a$  be the  $U(1)_a$  gauge field defined by

$$B_a = \text{tr } A_a. \quad (20)$$

For each  $U(N_a)$  factor, we define electric and magnetic charges of the colorless sector in the same way as the previous section. We denote the electric and magnetic  $U(1)_a$  charge by  $b_a$  and  $m_a$ , respectively. We collect them to form the vectors

$$\mathbf{b} = (b_1, \dots, b_n), \quad \mathbf{m} = (m_1, \dots, m_n), \quad (21)$$

and define the inner product

$$\mathbf{b} \cdot \mathbf{m} = \sum_{a=1}^n b_a m_a. \quad (22)$$

Each component of the vectors in (21) corresponds to each  $U(N_a)$  factor in the gauge group  $G$ , or, equivalently, each vertex in the quiver diagram, while each component of vectors  $\vec{w}$  or  $\vec{m}$  used in the last section corresponds to each  $U(1)$  factor in the Cartan subgroup of a single  $U(N)$ .

Colorless operators are constructed by combining

- Bi-fundamental chiral multiplets  $\Phi_I = (\phi_I, \psi_I)$
- $SU(N_a)$  invariant anti-symmetric tensors  $\epsilon_{(a)\alpha_1 \dots \alpha_{N_a}}$  and  $\epsilon_{(a)}^{\alpha_1 \dots \alpha_{N_a}}$
- Monopole operators  $\mathbf{m}[\mathbf{m}]$

We define charge matrix  $\{Q_{Ia}\}$  so that the component  $Q_{Ia}$  is +1 (−1) if  $\Phi_I$  belongs to the fundamental (anti-fundamental) representation of  $U(N_a)$ , and otherwise  $Q_{Ia} = 0$ .

The  $SU(N_a)$  index numbers, the electric and magnetic  $U(1)_a$  charges of the fundamental objects are shown in Table 2. We can again easily see that for



Table 2: The index numbers, the electric and magnetic charges are shown

	$z_a$	$b_a$	$m_a$
$\phi_I, \psi_I$	$Q_{Ia}$	$Q_{Ia}/N_a$	0
$\epsilon_{(a)}$	$-N_a$	0	0
$\mathbf{m}[\mathbf{m}]$	$k_a m_a$	$k_a m_a / N_a$	$m_a$

colorless operators the electric charge  $b_a$  is always an integer and is the same as the number of  $SU(N_a)$  invariant anti-symmetric tensor  $\epsilon_{(a)}$  included in the operator. Thus we can regard the charge  $b_a$  as the  $SU(N_a)$  baryon number. The complete contraction of color indices is possible only when the relation

$$N_a b_a - k_a m_a = z_a[\Phi] \quad (23)$$

holds, where  $z_a[\Phi]$  is the  $SU(N_a)$  index number carried by bi-fundamental fields in the operator. Because all matter fields in a quiver gauge theory are bi-fundamental fields, the right hand side in (23) vanishes when it is summed up with respect to  $a$ . We obtain

$$\mathbf{N} \cdot \mathbf{b} - \mathbf{k} \cdot \mathbf{m} = 0. \quad (24)$$

We define the charge lattice  $\Gamma$  of colorless operators as the set of vectors  $(\mathbf{b}, \mathbf{m})$  satisfying (24).

$$\Gamma = \{(\mathbf{b}, \mathbf{m}) | \mathbf{N} \cdot \mathbf{b} - \mathbf{k} \cdot \mathbf{m} = 0\} = \mathbb{Z}^{2n-1}. \quad (25)$$

To relate the charges  $(\mathbf{b}, \mathbf{m})$  and wrapping numbers of M-branes on the gravity side is a main purpose of this paper. In order to have clear geometric picture with wrapped branes, we take the large  $N$  limit. We assume that the ranks  $N_a$  are given by

$$N_a = N + \delta N_a, \quad (26)$$

and take the large  $N$  limit with  $\delta N_a$  fixed at order 1. We also assume that the charges  $b_a$  and  $m_a$  are of order 1.

$$b_a \sim \mathcal{O}(1), \quad m_a \sim \mathcal{O}(1). \quad (27)$$

If the charges are of order  $N$  the corresponding branes would deform the background geometry and the probe approximation would cease to be valid. Although it would be very interesting to investigate such a deformed geometry, we restrict ourselves to the case in which we can treat the branes as probes.

We separate operators into two classes, non-baryonic and baryonic operators. Non-baryonic operators are defined as operators with  $\mathbf{b} = 0$ . The other operators with  $\mathbf{b} \neq 0$  are referred to as baryonic operators. By definition, the non-baryonic operators are not only colorless but also gauge invariant. They are in general monopole operators carrying magnetic charges.

## 4 Monopoles and two-cycles

Let us first discuss correspondence between non-baryonic (monopole) operators and wrapped M2-branes. By definition, non-baryonic operators are characterized by only the magnetic charge  $\mathbf{m}$  constrained by

$$\mathbf{k} \cdot \mathbf{m} = 0. \quad (28)$$

The vector  $\mathbf{m}$  satisfying this condition spans the sublattice  $\Gamma_M \subset \Gamma$  defined by

$$\Gamma_M = \{(0, \mathbf{m}) | \mathbf{k} \cdot \mathbf{m} = 0\} = \mathbb{Z}^{n-1}. \quad (29)$$

We would like to relate monopole operators to wrapped M2-branes. However, it is known that a certain subset of these operators does not correspond to wrapped branes but to bulk Kaluza-Klein modes.

Let us temporarily consider the case with  $N_a = 1$ . We can regard this Abelian Chern-Simons theory as a subsector of the non-Abelian theory representing the motion of a single M2-brane. In the subsector, the Chern-Simons action (17) reduces to

$$S_{\text{CS}} = \sum_{a=1}^n \frac{k_a}{4\pi} \int A_a dA_a, \quad (30)$$

where  $A_a$  in this action should be interpreted as one of diagonal components of  $U(N_a)$  gauge field corresponding to the single M2-brane we are focusing on. Let us re-organize the  $n$   $U(1)$  gauge fields  $A_a$  into the diagonal  $U(1)$  gauge field  $A_D$  and the other  $n - 1$  gauge fields  $A'_i$  ( $i = 1, \dots, n - 1$ ). The relation between  $A_a$  and  $(A_D, A'_i)$  is

$$A_a = A_D + A'_a \quad (31)$$

where  $A'_a$  are linear combinations of  $A'_i$ . By substituting this into (30) we obtain

$$S_{\text{CS}} = \frac{1}{4\pi} \sum_{a=1}^n k_a \int A_D dA_D + \frac{1}{2\pi} \int A_D d \sum_{a=1}^n k_a A'_a + \sum_{a=1}^n \frac{k_a}{4\pi} \int A'_a dA'_a. \quad (32)$$

If we assume

$$\mathbf{1} \cdot \mathbf{k} = 0, \quad \mathbf{1} \equiv (1, 1, \dots, 1), \quad (33)$$

then the first term on the right hand side in (32) vanishes and the diagonal  $U(1)$  gauge field  $A_D$  appears in the action only through the second term in (32). The equation of motion of  $A_D$  is

$$d \sum_{a=1}^n k_a A_a = 0, \quad (34)$$

and we can solve this by

$$\sum_{a=1}^n k_a A_a = da. \quad (35)$$

The scalar field  $a$  is the dual-photon field. This is periodic scalar field with period  $2\pi$  and plays a role of the coordinate of the “eleventh” direction in the M-theory background. In the following, the relation (33) is always assumed because otherwise we cannot regard the theory as a theory of M2-branes.

Due to the periodicity of  $a$ , it is natural to define the operator  $e^{ia}$ . Because  $a$  is the canonical conjugate to the flux  $(2\pi)^{-1}dA_D$ , the operator  $e^{ia}$  changes the flux  $(2\pi)^{-1}dA_D$  by one. In other words, it carries the diagonal magnetic charge

$$\mathbf{m} = \mathbf{1}. \quad (36)$$

In the non-Abelian quiver gauge theory with gauge group (16), we should extend this operator to the monopole operator  $\mathbf{m}[\mathbf{1}]$  carrying the magnetic charge (36). By combining  $\mathbf{m}[\mathbf{1}]$  and the matter fields, we can always make colorless monopole operators with the same magnetic charge:

$$\mathcal{O} = \mathbf{m}[\mathbf{1}] \prod_I (\phi_I)^{s_I}. \quad (37)$$

The index numbers of  $\mathbf{m}[\mathbf{1}]$  are  $z_a = k_a$ , and for the operator (37) to be colorless,  $s_I$  must be integers solving the equation

$$k_a + \sum_I Q_{aI} s_I = 0 \quad \forall a. \quad (38)$$

If  $s_I$  is negative,  $(\phi_I)^{s_I}$  should be interpreted as  $(\phi_I^\dagger)^{-s_I}$ . Thanks to (33) and the connectivity of the quiver diagram, solutions always exist. If we would like to obtain chiral operators, we cannot use  $\phi_I^\dagger$  and  $s_I$  should be non-negative. Because we assume the theory is non-chiral and the quiver diagram is not only connected but also strongly connected, namely, every vertex is reachable from every other following oriented edges, the existence of such solutions is guaranteed.

In the correspondence between non-baryonic operators and wrapped M2-branes, we should exclude operators whose charges are multiple of (36). The exclusion of such operators is realized by introducing the equivalence relation

$$\mathbf{m} \sim \mathbf{m} + \mathbf{1}, \quad (39)$$

in the lattice  $\Gamma_M$ . We define the group of magnetic charges corresponding to wrapped M2-branes by

$$\Gamma_{M2} = \Gamma_M / (\mathbf{m} \sim \mathbf{m} + \mathbf{1}) = \mathbb{Z}^{n-2}. \quad (40)$$

Note that the constraint  $\mathbf{k} \cdot \mathbf{m} = 0$  and the equivalence relation  $\mathbf{m} \sim \mathbf{m} + \mathbf{1}$  are consistent to each other thanks to the assumption (33). We identify this group with the two-cycle homology of the internal space  $X_7$ :

$$H_2(X_7) = \Gamma_{M2} = \mathbb{Z}^{n-2}. \quad (41)$$

## 5 Baryons and five-cycles

Let us consider baryonic operators with  $\mathbf{b} \neq 0$ . We do not impose the condition  $\mathbf{m} = 0$  for baryonic operators, and in general baryonic operator may carry magnetic charges. We define the group of baryonic charges by neglecting the charges of monopole operators. Namely, we define the charge lattice of baryonic operators as the following quotient lattice:

$$\Gamma_B = \Gamma / \Gamma_M. \quad (42)$$

The constraint (24) gives

$$\mathbf{N} \cdot \mathbf{b} = 0 \pmod{\gcd \mathbf{k}}. \quad (43)$$

In the large  $N$  limit, by using (26) and (27), we decompose this into the conditions

$$\mathbf{1} \cdot \mathbf{b} = 0, \quad (44)$$

and

$$\delta \mathbf{N} \cdot \mathbf{b} = 0 \pmod{\gcd \mathbf{k}}. \quad (45)$$

The first condition (44) guarantees that (24) can be satisfied with  $\mathbf{m}$  of order 1. This condition is necessary because we only consider operators realized on the gravity side as probe branes. When (44) is satisfied, (43) becomes the constraint (45), which will be regarded as the condition for the absence of flux strings attached on the operator.

Although the condition (45) must hold for the operator to be colorless, it is convenient to define the lattice  $\Gamma'_B$  defined only by the first constraint (44).

$$\Gamma'_B = \{\mathbf{b} \in \mathbb{Z}^n | \mathbf{1} \cdot \mathbf{b} = 0\} = \mathbb{Z}^{n-1}. \quad (46)$$

A vector in  $\Gamma'_B$  in general gives colored operators accompanied by flux strings, and the second condition (45) defines the lattice  $\Gamma_B$  of colorless operators as a sublattice of  $\Gamma'_B$ .

Similarly to the case of non-baryonic operators, a certain subset of  $\Gamma_B$  does not correspond to wrapped M5-branes. Let us consider  $\delta \mathbf{N} = 0$  case first. In this case, we can define the dual-photon field in the non-Abelian theory in the same way as the Abelian ( $N_a = 1$ ) case. The dual photon field is defined by

$$da = \sum_{a=1}^n k_a B_a \quad (47)$$

where  $B_a$ , which is defined in (20), is the gauge fields coupling to the charge  $b_a$ . Under gauge transformation  $\delta B_a = d\lambda_a$ , the dual photon field is transformed by

$$\delta a = \sum_{a=1}^n k_a \lambda_a. \quad (48)$$

Due to this non-linear gauge transformation, the expectation value of the dual photon field breaks a  $U(1)$  subgroup of  $G_B$  into a certain discrete group. Therefore, the charge associated with this broken  $U(1)$  is no longer conserved, and cannot be identified with any wrapping number of M5-branes on the gravity side. Thus, to remove this unconserved component from the charges, we introduce the equivalence relation

$$\mathbf{b} \sim \mathbf{b} + \mathbf{k}, \quad (49)$$

representing the “screening” by the operator  $e^{ia}$ , and define the baryonic charge group by

$$\Gamma'_{M5} = \Gamma'_B / (\mathbf{b} \sim \mathbf{b} + \mathbf{k}) = \mathbb{Z}^{n-2} \times \mathbb{Z}_{\gcd \mathbf{k}}. \quad (50)$$

Note that if  $\delta \mathbf{N} = 0$  (45) is automatically satisfied and  $\Gamma'_B = \Gamma_B$ . We identify the group (50) with the five-cycle homology group  $H_5(X_7)$ .

Let us next consider the general case with  $\delta \mathbf{N} \neq 0$ . Even in this case the topology of the internal space is expected not to change from the case of  $\delta \mathbf{N} = 0$  as long as  $\delta \mathbf{N}$  is of order one, and we still identify the group (50) with the five-cycle homology.

$$H_5(X_7) = \Gamma'_{M5} = \mathbb{Z}^{n-2} \times \mathbb{Z}_{\gcd \mathbf{k}}. \quad (51)$$

It is, however, not necessarily the same as the group of isolated wrapped M5-branes because wrapped M5-branes are in general accompanied by flux strings realized as wrapped M2-branes as is studied in more detail in the next section. We define the group of wrapped M5-branes without flux strings as a subset of  $\Gamma'_{M5}$  by requiring the condition (45).

$$\Gamma_{M5} = \{[\mathbf{b}] \in \Gamma'_{M5} | \delta \mathbf{N} \cdot \mathbf{b} = 0 \pmod{\gcd \mathbf{k}}\}, \quad (52)$$

where  $[\mathbf{b}] = \mathbf{b} + \mathbb{Z}\mathbf{k}$  is the equivalence class with representative  $\mathbf{b}$ . The inner product in (52) as an element of  $\mathbb{Z}_{\gcd \mathbf{k}}$  does not depend on the choice of a representative from  $[\mathbf{b}]$  because  $\delta \mathbf{N} \cdot \mathbf{k} = 0 \pmod{\gcd \mathbf{k}}$ .

The combination of two conditions (44) and (45) is equivalent to the single condition (43) only under the restriction (27). If we permit magnetic charge of order  $N$ , there exist colorless operators whose charge  $\mathbf{b}$  satisfies (43), but not (44) and (45) separately. An example of such operators is the following monopole operator associated with a single  $U(N)$  gauge group:

$$\epsilon_{\alpha_1 \dots \alpha_N} \mathbf{m}[\vec{m}]^{\alpha_1 \dots \alpha_N}, \quad \vec{m} = (1, 1, \dots, 1). \quad (53)$$

This operator, however, is prohibited by a gauge invariance condition as we explain below. The reason why we have not imposed gauge invariance with respect to the  $U(1)$  part of  $U(N)$  groups is that some of gauge fields of these  $U(1)$  are regarded as the boundary values of bulk gauge fields, and couple to wrapped M-branes [41]. If such a bulk gauge field is absent for a  $U(1)$  gauge symmetry on the

boundary, the gauge invariance with respect to this  $U(1)$  must be imposed. Once we accept the relation (51), we have only  $b_5 = n - 1$  bulk gauge fields coupling to wrapped M5-branes, and no bulk gauge field couples the diagonal baryonic charge  $\mathbf{1} \cdot \mathbf{b}$ . Therefore, concerning this diagonal part, we must impose the gauge invariance condition, which is nothing but (24). Thus, the operator (53) does not have its counterpart on the gravity side.

## 6 Flux strings and ranks of gauge groups

In general if we introduce an external source of the color charge, the charge is partially screened by ambient fields. Well-known example is that the color charge in the  $SU(N)$  pure Yang-Mills theory is screened by the adjoint field to leave only the “ $N$ -aliy” of the representation. In a confining theory, an external source with unscreened charge is accompanied by a flux string. This is not the case in non-confining theories. Even in such non-confining theories, on the gravity side, operators with unscreened charge is treated as endpoints of stringy objects in AdS space. For example, external quarks in the maximally supersymmetric Yang-Mills theory in four dimensions are treated as the endpoints of fundamental strings on the conformal boundary [45, 46]. We will use the term “flux strings” in the following to mean such stringy objects in  $AdS_4$ .

In this section we treat baryonic operators as external sources, and discuss flux strings attached on them. What degrees of freedom is left after screening in a quiver Chern-Simons theory? If we take account of the vacuum polarization of adjoint fields, the information of a  $U(N_a)$  representation is almost lost and we are left with only the index number  $z_a$  for each  $U(N_a)$ . The polarization of bi-fundamental fields hides the distinction among  $U(N_a)$  factors, and only the total index number

$$z = \sum_{a=1}^n z_a \quad (54)$$

is left. If we take account of all non-baryonic operators including monopole operators, only the modulo  $\gcd \mathbf{k}$  part of  $z$  is left unscreened because as is shown in Table 2 the index number of monopole operators is linear combination of Chern-Simons levels  $k_a$  with integral coefficients. If this unscreened charge does not vanish, the source is accompanied by flux strings.

In the previous section, we saw that the electric charge  $\mathbf{b}$  of a colorless baryonic operator satisfies (45). We can regard this as the condition for the complete screening of the charges of the operator. If (45) does not hold, the baryonic operator is accompanied by a flux string with charge

$$f = \delta \mathbf{N} \cdot \mathbf{b} \in \mathbb{Z}_{\gcd \mathbf{k}}. \quad (55)$$

On the gravity side, we can interpret this relation as follows. Let us consider a baryonic operator realized as an M5-brane wrapped on a five-cycle  $\Omega_5$ . The

action of the M5-brane includes

$$\frac{1}{2\pi} \oint_{M5} H_3 \wedge C_3 = \frac{1}{2\pi} \oint_{M5} b_2 \wedge F_4, \quad (56)$$

where  $H_3 = db_2$  is the field strength of the two-form field  $b_2$  living on the M5-brane, and  $F_4 = dC_3$  is the field strength of the background three-form field  $C_3$ . Let  $\Sigma_3 \in H_3(X_7)$  be the Poincare dual of the background four-form flux  $[(2\pi)^{-1}F_4] \in H^4(X_7)$ . Through the interaction (56), the background flux induces the charge on the M5-brane worldvolume electrically coupled by the field  $b_2$ . Because  $\Omega_5$  is compact, the charge must be canceled by the charge of the boundary of M2-branes attached on the M5-brane. For this cancellation we need to attach M2-brane along the one-cycle  $\gamma_1$  in  $\Omega_5$  which is Poincare dual in  $\Omega_5$  to

$$\left[ \frac{1}{2\pi} F_4|_{\Omega_5} \right] \in H^4(\Omega_5). \quad (57)$$

In other words,  $\gamma_1$  is the intersection of the five-cycle  $\Omega_5$  and the three-cycle  $\Sigma_3$

$$\gamma_1 = \Omega_5 \cap \Sigma_3. \quad (58)$$

This is the geometric translation of the relation (55). We identify flux strings with M2-branes wrapped on one-cycles, and the flux string charge group with the one-cycle homology

$$H_1(X_7) = \mathbb{Z}_{\text{gcd } \mathbf{k}}. \quad (59)$$

Flux strings generate non-trivial monodromies for baryonic operators (wrapped M5-branes).

Up to now, we have obtained the following homologies by the comparison of operators in a Chern-Simons theory and their M-brane realizations:

$$H_1(X_7) = \mathbb{Z}_{\text{gcd } \mathbf{k}}, \quad H_2(X_7) = \mathbb{Z}^{n-2}, \quad H_5(X_7) = \mathbb{Z}^{n-2} \times \mathbb{Z}_{\text{gcd } \mathbf{k}}. \quad (60)$$

These are consistent to the duality of the homology groups. In general, the following duality relations hold among homologies of  $d$ -dimensional manifold:

$$H_i^f = H_{d-i}^f, \quad H_i^t = H_{d-i-1}^t, \quad (61)$$

where  $H_i^f$  and  $H_i^t$  are the free part and the torsion subgroup, respectively, of the homology  $H_i$ .

In the above argument, we relate the three-cycle homology  $H_3(X_7)$  to  $\delta \mathbf{N}$ , the “fractional” part of the ranks:

$$\text{three-cycles} \quad \leftrightarrow \quad \delta \mathbf{N}. \quad (62)$$

In general, the structure of the three-cycle homology is highly non-trivial, and we do not try to establish the concrete map between three-cycles and  $\delta \mathbf{N}$ . We here

comment on one important point;  $\delta\mathbf{N} = 0$  does not necessarily mean vanishing four-form flux. If all the ranks are the same and  $\delta\mathbf{N} = 0$ , (45) is automatically satisfied. On the gravity side, this means that the three-cycle  $\Sigma_3$  satisfies

$$[\Omega_5 \cap \Sigma_3] = [0] \in H_1(X_7) \quad \forall [\Omega_5] \in H_5(X_7). \quad (63)$$

The condition (63) does not require  $[\Sigma_3] = 0$ , the vanishing background four-form flux. Indeed, in the case of  $\mathcal{N} = 4$  Chern-Simons theories, there are in general many possible  $F_4$  discrete torsion corresponding to equal-rank quiver Chern-Simons theories [41]. We will mention such an example in the following section.

In addition to M2-branes wrapped on one-cycles, there is another potential origin of stringy objects: M5-branes wrapped on four-cycles. Combining the two-cycle homology (41) and the duality relation (61), we find that  $H_4(X_7)$  does not have torsion subgroup. The duality relation also says that it is the same as the free part of  $H_3(X_7)$ .

$$H_4(X_7) = H_3^f(X_7) = \mathbb{Z}^{b_3}. \quad (64)$$

Absence of the torsion subgroup in  $H_4(X_7)$  means that associated strings does not induce fractional monodromies for monopole operators. We have no idea about interpretation of these strings. We only comment that these may have something to do with the cascading phenomenon. If  $b_3 \neq 0$ , we can introduce four-form flux in the free part of the four-form cohomology group  $H^4(X_7) = H_3(X_7)$ . Unlike the discrete torsion, such a flux induces non-vanishing energy and deforms the background geometry. Such a deformation signals the existence of cascading phenomenon [37] in three dimensions.

The homologies we obtained up to now are collected in Table 3. They are completely determined by three integers,  $s$ ,  $b_2$ , and  $b_3$ , and torsion part  $T$  of  $H_3(X_7)$ . The integers  $s$  and  $b_2$  are given in terms of parameters in the Chern-

Table 3: Homologies of  $X_7$  conjectured from the charge analysis in quiver Chern-Simons theories.

	$H_0$	$H_1$	$H_2$	$H_3$	$H_4$	$H_5$	$H_6$	$H_7$
free part	$\mathbb{Z}$	0	$\mathbb{Z}^{b_2}$	$\mathbb{Z}^{b_3}$	$\mathbb{Z}^{b_3}$	$\mathbb{Z}^{b_2}$	0	$\mathbb{Z}$
torsion part	0	$\mathbb{Z}_s$	0	$T$	0	$\mathbb{Z}_s$	0	0

Simons theory by

$$s = \gcd \mathbf{k}, \quad b_2 = n - 2. \quad (65)$$



## 7 Examples

In the previous sections, by the analysis of charges in non-chiral Chern-Simons theories, we conjectured the homology groups of the dual geometry  $X_7$  as Table 3. Typical examples of Sasaki-Einstein manifolds/orbifolds are listed in Table 4. All

Table 4: The data  $s$ ,  $b_2$ ,  $b_3$ , and  $T$  of various Einstein manifolds/orbifolds are shown. Refer to the indicated references for  $T$  of the last four manifolds/orbifolds. (For  $V^{5,2}$  only the fact that  $H_3$  is at most torsion is mentioned in [47].)

$X_7$	$s$	$b_2$	$b_3$	$T$
$\mathbf{S}^7/\mathbb{Z}_k$ [11]	$k$	0	0	$\mathbb{Z}_k$
$Q^{1,1,1}$ [47]	1	2	0	$\mathbb{Z}_2$
$M^{1,1,1}$ [47]	1	1	0	$\mathbb{Z}_9$
$N^{0,1,0}$ [47]	1	1	0	0
$V^{5,2}$ [47]	1	0	0	*
$Y^{p,k}(\mathbb{C}P^2)$ [48]	$\gcd(p, k)$	1	0	*
$Y^{p,k}(\mathbb{C}P^1 \times \mathbb{C}P^1)$ [48]	$\gcd(p, k)$	2	0	*
$(\mathbf{S}^7/(\mathbb{Z}_p \times \mathbb{Z}_q))/\mathbb{Z}_k$ [41]	$k$	$p + q - 2$	0	*

examples in the table have vanishing  $b_3$ . It is also known that every 3-Sasakian manifold has  $b_3 = 0$ . This is not the case for general Einstein manifolds. The simplest example is  $\mathbf{S}^3 \times \mathbf{S}^4$  with appropriate radii. Sasaki-Einstein manifolds with  $b_3 \neq 0$  are also known to exist. (See [49] and references therein.) Although all manifolds/orbifolds in the table have homologies consistent with the form shown in Table 3, only the first and the last examples correspond to non-chiral theories, and our arguments are not applicable to the others.

In the following subsections, we discuss the two examples of non-chiral theories in more detail.

### 7.1 ABJM model

The simplest example is the  $\mathcal{N} = 6$  Chern-Simons theory proposed by Aharony, Bergman, Jafferis, and Maldacena [11]. This model, ABJM model, is described by a quiver diagram with two vertices. Namely, the gauge group is  $U(N_1) \times U(N_2)$ . Due to the condition (33) two  $U(N)$  gauge groups have opposite Chern-Simons levels.  $n$  and  $\mathbf{k}$  are given by

$$n = 2, \quad \mathbf{k} = (k, -k). \quad (66)$$

The Higgs branch moduli space of this theory is the symmetric product of  $\mathbb{C}^4/\mathbb{Z}_k$ , and the internal space is  $X_7 = \mathbf{S}^7/\mathbb{Z}_k$ . (66) is consistent through (65) with the

data of this internal space shown in Table 4. The non-trivial homologies are

$$H_0 = H_7 = \mathbb{Z}, \quad H_1 = H_3 = H_5 = \mathbb{Z}_k. \quad (67)$$

Let  $\sigma_3$  and  $\sigma_5$  be the generators of  $H_3$  and  $H_5$ , respectively. We can adopt  $\sigma_1 = \sigma_3 \cap \sigma_5$ , the intersection of  $\sigma_3$  and  $\sigma_5$ , as the generator of the one-cycle homology group.

There are no non-trivial two-cycles in  $X_7 = \mathbf{S}^7/\mathbb{Z}_k$ . Monopole operators in ABJM model [50, 51, 52] carry “diagonal” magnetic charge proportional to  $\mathbf{1} = (1, 1)$ , and should be identified with Kaluza-Klein modes in the bulk.

Baryonic operators in ABJM model with  $\delta\mathbf{N} = 0$  are studied in [53], and the degeneracy and the conformal dimension are reproduced by the analysis using wrapped branes. (In [53] the wrapped branes are analyzed from the perspective of type IIA theory.)

The gravity dual of  $U(N) \times U(N+M)$  ABJM model is studied in [54]. The rank difference  $M$  in this case correspond to the fractional brane wrapped on  $\Sigma_3 = M\sigma_3$ , or, equivalently, the discrete torsion  $F_4 = \Sigma_3^*$ , where  $\Sigma_3^*$  is the Poincare dual of  $\Sigma_3$ . In [54] it is argued that only when  $-k \leq M \leq k$  the Chern-Simons theory is unitary, and two theories with  $M = M_1$  and  $M = M_2$  are equivalent if  $M_1 = M_2 \pmod k$ .

When  $0 \leq M \leq k$ , we can construct the following baryonic operator:

$$\mathcal{B}^{\beta_{N+1} \cdots \beta_{N+M}} = \epsilon_{\alpha_1 \cdots \alpha_N} \epsilon^{\beta_1 \cdots \beta_N \beta_{N+1} \cdots \beta_{N+M}} \phi_{\beta_1}^{\alpha_1} \phi_{\beta_2}^{\alpha_2} \cdots \phi_{\beta_N}^{\alpha_N}, \quad (68)$$

where  $\alpha_i$  and  $\beta_i$  are  $SU(N)$  and  $SU(N+M)$  indices, respectively, and  $\phi_\beta^\alpha$  is bi-fundamental scalar field. ABJM model includes four such bi-fundamental scalar fields. In (68) we omitted the flavor indices for distinction of these four. The operator (68) correspond to an M5-brane wrapped on the five-cycle  $\sigma_5$ . If  $1 \leq M \leq k-1$ , this is not colorless, and is accompanied by a flux string with charge  $M \in \mathbb{Z}_k$ . Geometric description (58) of this fact is

$$\Sigma_3 \cap \sigma_5 = M\sigma_1. \quad (69)$$

If  $M = 0$ , (68) is colorless, and not accompanied by a flux string. If two theories with  $M = 0$  and  $M = k$  are equivalent as is argued in [54], it should be possible to construct colorless baryonic operator even when  $M = k$ . Actually, this is possible. By adding fermionic bi-fundamental fields  $\psi_\beta^\alpha$  and the monopole operator  $\mathfrak{m}[(-1, 0)]$  to (68), we can write the colorless operator

$$\mathcal{B} = \epsilon_{\alpha_1 \cdots \alpha_N} \mathfrak{m}[(-1, 0)]_{\alpha_{N+1} \cdots \alpha_{N+k}} \epsilon^{\beta_1 \cdots \beta_N \beta_{N+1} \cdots \beta_{N+k}} \phi_{\beta_1}^{\alpha_1} \phi_{\beta_2}^{\alpha_2} \cdots \phi_{\beta_N}^{\alpha_N} \psi_{\beta_{N+1}}^{\alpha_{N+1}} \cdots \psi_{\beta_{N+k}}^{\alpha_{N+k}}. \quad (70)$$

## 7.2 $\mathcal{N} = 4$ Chern-Simons theories

$\mathcal{N} = 4$  Chern-Simons theories are described by circular quiver diagrams whose vertices and edges represent  $U(N)$  gauge groups and hyper multiplets, respectively. There are two types of hypermultiplets, so-called untwisted and twisted

hypermultiplets [10]. Let  $n$ ,  $p$ , and  $q$  be the number of vector, untwisted hyper, and twisted hypermultiplets, respectively. Because the quiver diagram is circular, the following relation holds:

$$p + q = n. \quad (71)$$

The requirement of  $\mathcal{N} = 4$  supersymmetry restricts the Chern-Simons levels to be  $\pm k$  or 0. The Higgs branch moduli space of this theory is derived in [55]. See also [56, 57]. It is the symmetric product of

$$(\mathbb{C}^2/\mathbb{Z}_p \times \mathbb{C}^2/\mathbb{Z}_q)/\mathbb{Z}_k. \quad (72)$$

Correspondingly, the internal space is

$$X_7 = (\mathbf{S}^7/(\mathbb{Z}_p \times \mathbb{Z}_q))/\mathbb{Z}_k. \quad (73)$$

The data of the homology groups of this orbifold are given in Table 4, and satisfy the relation (65). In [41] not only the isomorphisms  $\Gamma_{M2} = H_2(X_7)$  and  $\Gamma'_{M5} = H_5(X_7)$  but also the agreement of degeneracy and the conformal dimension of baryonic operators with the predictions of the M5-brane description is confirmed. Concerning monopole operators, the R-charge spectrum are computed on the field theory side in [58] by using the radial quantization method [42, 43], and agreement with the analysis on the gravity side is partially confirmed.

An interesting feature of these theories is the non-trivial structure of  $H_3(X_7)$ . It is given by

$$H_3(X_7) = (\mathbb{Z}_{kp}^{q-1} \times \mathbb{Z}_{kq}^{p-1} \times \mathbb{Z}_{kpq})/(\mathbb{Z}_p \times \mathbb{Z}_q). \quad (74)$$

Refer to [41] for detailed description of this homology group. To understand the meaning of  $H_3(X_7)$  on the field theory side, it is convenient to realize the theory by type IIB brane system. When  $k = 1$ , the theory is realized on a system consisting of D3-branes wrapped around  $\mathbf{S}^1$ , on which gauge theory lives, and  $p$  NS5 and  $q$  D5-branes intersecting with the D3-branes. The fivebranes divide the  $\mathbf{S}^1$  into  $n$  intervals. Here, we discuss only the case with  $p = q = 2$  and  $k = 1$  for simplicity and concreteness. In this case, the non-trivial homologies are

$$H_0 = H_7 = \mathbb{Z}, \quad H_2 = H_5 = \mathbb{Z}^2, \quad H_3 = \mathbb{Z}_4. \quad (75)$$

In order to specify the brane configuration, we need to specify the arrangement of the four fivebranes. For this purpose, we decorate the rank vector in the following way:

$$\mathbf{N} = ({}^{1:NS} N_1, {}^{2:NS} N_2, {}^{3:D} N_3, {}^{4:D} N_4, {}^{1:NS}). \quad (76)$$

The superscripts in (76) mean that the fivebranes are arranged along  $\mathbf{S}^1$  in order NS5, NS5, D5, and D5. The gauge group  $U(N_i)$  corresponding to  $i$ -th component of the vector  $\mathbf{N}$  is realized on  $N_i$  D3-branes stretched between two fivebranes indicated on the two sides of the component  $N_i$  in the vector (76). For the brane configuration represented in (76), the levels are

$$\mathbf{k} = (0, 1, 0, -1). \quad (77)$$

The Chern-Simons level of each  $U(N)$  depends on the fivebranes at the two ends of the interval, and we can read off the rule to determine Chern-Simons levels for general ordering of fivebranes from this example.

Let  $\sigma_3$  be the generator of  $H_3 = \mathbb{Z}_4$ . The relation between the  $F_4$  discrete torsion and the structure of the brane system is investigated in [41]. We can also obtain some information about this relation from the analysis of the monopole spectrum in [58]. Results in these references indicate that the rank vector of the Chern-Simons theory corresponding to the discrete torsion  $F_4 = M\sigma_3^*$  is

$$\mathbf{N} = ({}^{1:NS}N, {}^{2:NS}N + M, {}^{3:D}N, {}^{4:D}N, {}^{1:NS}N). \quad (78)$$

This is only one of infinitely many possible choice of the brane configuration, which are transformed to one another by continuous interchanges of fivebranes. Such deformations are expected to have something to do with Seiberg-like duality in three dimensions [59, 60].

The rank vector (78) may seem to show that the equal-rank gauge group is realized only when the discrete torsion vanishes. This is, however, not a precise statement because even if  $M \neq 0$  it may be possible to realize equal ranks by interchanges of fivebranes. Actually, it is possible when  $M = 0, \pm 1 \pmod{4}$ . In the case of  $M = 1$ , we can realize equal ranks by exchanging the fivebranes 2 and 3.

$$({}^{1:NS}N, {}^{2:NS}N + 1, {}^{3:D}N, {}^{4:D}N, {}^{1:NS}N) \xrightarrow{[23]} ({}^{1:NS}N, {}^{2:D}N, {}^{3:NS}N, {}^{4:D}N, {}^{1:NS}N). \quad (79)$$

We took account of the brane creation due to the Hanany-Witten effect [61]. The Chern-Simons level for the resulting brane configuration is  $\mathbf{k} = (1, -1, 1, -1)$ . When  $M = -1$ , we can realize equal ranks by three steps as follows.

$$\begin{aligned} & ({}^{1:NS}N, {}^{2:NS}N - 1, {}^{3:D}N, {}^{4:D}N, {}^{1:NS}N) \\ & \xrightarrow{[12]} \xrightarrow{[34]} \xrightarrow{[14]} ({}^{1:D}N - 1, {}^{2:NS}N - 1, {}^{3:D}N - 1, {}^{4:NS}N - 1, {}^{1:D}N). \end{aligned} \quad (80)$$

The levels for the resulting brane configuration is  $\mathbf{k} = (-1, 1, -1, 1)$ . Similar deformation to equal rank configuration is always possible if  $M = 0, \pm 1 \pmod{4}$ . See [41] for detailed analysis of such brane interchange processes. If the gauge group is equal-rank, the condition (45) is trivially satisfied, and we can define a colorless baryonic operator corresponding to any five-cycle in  $X_7$ .

If we start from the rank vector (78) with  $M = 2 \pmod{4}$ , we cannot arrive at any equal-rank configuration. Even in this case, the condition (45) still holds with  $k = 1$  and it should be possible to construct a colorless baryonic operator for an arbitrary five-cycle. Let us consider, for example, a baryonic operator with charge  $\mathbf{b} = (-1, 1, 0, 0)$  in the theory with

$$\mathbf{N} = ({}^{1:NS}N, {}^{2:NS}N + 2, {}^{3:D}N, {}^{4:D}N, {}^{1:NS}N), \quad \mathbf{k} = (0, 1, 0, -1). \quad (81)$$

We can indeed construct the colorless operator

$$\mathcal{B} = \epsilon^{\alpha_1 \dots \alpha_N} \mathbf{m}^{\beta_{N+1} \beta_{N+2}} \epsilon_{\beta_1 \dots \beta_N \beta_{N+1} \beta_{N+2}} \phi_{\alpha_1}^{\beta_1} \phi_{\alpha_2}^{\beta_2} \dots \phi_{\alpha_N}^{\beta_N}, \quad (82)$$

where  $\alpha_i$  and  $\beta_i$  are  $U(N_1)$ , and  $U(N_2)$  color indices, respectively.  $\mathbf{m}$  is a monopole operator with magnetic charge  $(0, 2, 0, 0)$ .

## 8 Discussions

In this paper we considered following aspects in non-chiral  $\mathcal{N} = 2$  quiver Chern-Simons theories:

- We defined the lattice  $\Gamma_{M2}$  of magnetic charges of non-baryonic operators, and identified it with the two-cycle homology  $H_2(X_7)$  of the internal space  $X_7$ .
- The lattice of baryonic charge  $\Gamma'_{M5}$  were defined. The colorless baryonic operators forms the sublattice  $\Gamma_{M5} \subset \Gamma'_{M5}$ . The former was identified with the five-cycle homology  $H_5(X_7)$ .
- The charge of flux strings  $\mathbb{Z}_{\text{gcd } \mathbf{k}}$  were identified with the one-cycle homology  $H_1(X_7)$ .
- The charge of flux strings attached on baryonic operators depends on the ranks of gauge groups, and is obtained by the relation (55). We derived the corresponding relation (58) on the gravity side by requiring the flux conservation on M5-branes. We can use these relation to obtain some information about the relation between ranks of gauge groups and the four-form flux in the dual geometry.

There are many problems which we did not study in this paper.

The one-to-one correspondence between operators and dual objects in  $\text{AdS}_5/\text{CFT}_4$  has been intensively investigated. In particular, in the maximally supersymmetric Yang-Mills theory in four-dimensions, the duality between 1/2 BPS operators classified by Schur polynomials [62, 63] and giant gravitons [64, 65] or bubbling geometries [66] was found. It is interesting problem to establish a similar one-to-one correspondence between operators and objects in the dual geometry in the case of  $\text{AdS}_4/\text{CFT}_3$ .

The three-cycle homology group  $H_3(X_7)$  is expected to relate to the rank distribution in the gauge group. In general, the structure of  $H_3$  is complicated, and it is not straightforward to establish the map between  $\delta \mathbf{N}$  and elements of  $H_3$ . The relation (58), which corresponds to (55) in the field theory, gives some information. This is, however, not sufficient to establish the complete map. More detailed information may be obtained by analyzing the spectrum of monopole

operators. Because the three-form potential couples to M2-branes, it works as Wilson lines shifting the Kaluza-Klein spectrum of wrapped M2-branes. By comparing spectrum of monopole operators and that of wrapped M2-branes we can obtain additional information about the relation (62). Spectrum of monopole operators is studied in [58] for  $\mathcal{N} = 4$  Abelian Chern-Simons theories. Extension of this analysis to more general theories is important task.

Generalization of our results to chiral theories is very interesting and challenging problem. It would enable us to consider a much larger class of Chern-Simons theories and Sasaki-Einstein dual geometries, such as dual pairs constructed by utilizing brane crystals [67, 68, 69], and might provide information about dynamics of Chern-Simons theories with large quantum corrections.

We hope to return to these problems in the near future.

## Acknowledgements

I would like to thank Y. Yasui for valuable information about Einstein manifolds. I also would like to thank S. Yokoyama for discussions. This work was supported in part by Grant-in-Aid for Young Scientists (B) (#19740122) from the Japan Ministry of Education, Culture, Sports, Science and Technology.

## References

- [1] J. Bagger and N. Lambert, Phys. Rev. D **75**, 045020 (2007) [arXiv:hep-th/0611108].
- [2] J. Bagger and N. Lambert, Phys. Rev. D **77**, 065008 (2008) [arXiv:0711.0955 [hep-th]].
- [3] J. Bagger and N. Lambert, JHEP **0802**, 105 (2008) [arXiv:0712.3738 [hep-th]].
- [4] A. Gustavsson, arXiv:0709.1260 [hep-th].
- [5] A. Gustavsson, JHEP **0804**, 083 (2008) [arXiv:0802.3456 [hep-th]].
- [6] N. Lambert and D. Tong, Phys. Rev. Lett. **101**, 041602 (2008) [arXiv:0804.1114 [hep-th]].
- [7] J. Distler, S. Mukhi, C. Papageorgakis and M. Van Raamsdonk, JHEP **0805**, 038 (2008) [arXiv:0804.1256 [hep-th]].
- [8] D. Gaiotto and E. Witten, arXiv:0804.2907 [hep-th].

- [9] H. Fuji, S. Terashima and M. Yamazaki, Nucl. Phys. B **810**, 354 (2009) [arXiv:0805.1997 [hep-th]].
- [10] K. Hosomichi, K. M. Lee, S. Lee, S. Lee and J. Park, JHEP **0807**, 091 (2008) [arXiv:0805.3662 [hep-th]].
- [11] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, arXiv:0806.1218 [hep-th].
- [12] K. Hosomichi, K. M. Lee, S. Lee, S. Lee and J. Park, JHEP **0809**, 002 (2008) [arXiv:0806.4977 [hep-th]].
- [13] J. Bagger and N. Lambert, arXiv:0807.0163 [hep-th].
- [14] M. Schnabl and Y. Tachikawa, arXiv:0807.1102 [hep-th].
- [15] Y. Imamura and K. Kimura, J. High Energy Phys. **10** (2008) 040, arXiv:0807.2144.
- [16] H. Ooguri and C. S. Park, JHEP **0811**, 082 (2008) [arXiv:0808.0500 [hep-th]].
- [17] D. L. Jafferis and A. Tomasiello, JHEP **0810**, 101 (2008) [arXiv:0808.0864 [hep-th]].
- [18] D. Martelli and J. Sparks, arXiv:0808.0912 [hep-th].
- [19] A. Hanany and A. Zaffaroni, arXiv:0808.1244 [hep-th].
- [20] K. Ueda and M. Yamazaki, arXiv:0808.3768 [hep-th].
- [21] Y. Imamura and K. Kimura, JHEP **0810**, 114 (2008) [arXiv:0808.4155 [hep-th]].
- [22] A. Hanany, D. Vegh and A. Zaffaroni, arXiv:0809.1440 [hep-th].
- [23] S. Franco, A. Hanany, J. Park and D. Rodriguez-Gomez, JHEP **0812**, 110 (2008) [arXiv:0809.3237 [hep-th]].
- [24] A. Hanany and Y. H. He, arXiv:0811.4044 [hep-th].
- [25] S. Hohenegger and I. Kirsch, arXiv:0903.1730 [hep-th].
- [26] D. Gaiotto and D. L. Jafferis, arXiv:0903.2175 [hep-th].
- [27] E. Witten, JHEP **9807**, 006 (1998) [arXiv:hep-th/9805112].
- [28] S. S. Gubser and I. R. Klebanov, Phys. Rev. D **58**, 125025 (1998) [arXiv:hep-th/9808075].

- [29] S. Gukov, M. Rangamani and E. Witten, JHEP **9812**, 025 (1998) [arXiv:hep-th/9811048].
- [30] A. Butti, D. Forcella and A. Zaffaroni, JHEP **0706**, 069 (2007) [arXiv:hep-th/0611229].
- [31] D. Forcella, A. Hanany and A. Zaffaroni, JHEP **0712**, 022 (2007) [arXiv:hep-th/0701236].
- [32] A. Butti, D. Forcella, A. Hanany, D. Vegh and A. Zaffaroni, JHEP **0711**, 092 (2007) [arXiv:0705.2771 [hep-th]].
- [33] A. Hanany and K. D. Kennaway, “Dimer models and toric diagrams,” arXiv:hep-th/0503149.
- [34] S. Franco, A. Hanany, K. D. Kennaway, D. Vegh and B. Wecht, JHEP **0601** (2006) 096, arXiv:hep-th/0504110.
- [35] S. Franco, A. Hanany, D. Martelli, J. Sparks, D. Vegh and B. Wecht, JHEP **0601** (2006) 128, arXiv:hep-th/0505211.
- [36] I. R. Klebanov and A. A. Tseytlin, Nucl. Phys. B **578**, 123 (2000) [arXiv:hep-th/0002159].
- [37] I. R. Klebanov and M. J. Strassler, JHEP **0008**, 052 (2000) [arXiv:hep-th/0007191].
- [38] I. R. Klebanov and E. Witten, Nucl. Phys. B **536**, 199 (1998) [arXiv:hep-th/9807080].
- [39] S. Benvenuti, A. Hanany and P. Kazakopoulos, JHEP **0507**, 021 (2005) [arXiv:hep-th/0412279].
- [40] A. Butti, JHEP **0610**, 080 (2006) [arXiv:hep-th/0603253].
- [41] Y. Imamura and S. Yokoyama, arXiv:0812.1331(v3) [hep-th].
- [42] V. Borokhov, A. Kapustin and X. k. Wu, JHEP **0211**, 049 (2002) [arXiv:hep-th/0206054].
- [43] V. Borokhov, A. Kapustin and X. k. Wu, JHEP **0212**, 044 (2002) [arXiv:hep-th/0207074].
- [44] P. Goddard, J. Nuyts and D. I. Olive, “Gauge Theories And Magnetic Charge,” Nucl. Phys. B **125**, 1 (1977).
- [45] S. J. Rey and J. T. Yee, Eur. Phys. J. C **22**, 379 (2001) [arXiv:hep-th/9803001].



- [46] J. M. Maldacena, Phys. Rev. Lett. **80**, 4859 (1998) [arXiv:hep-th/9803002].
- [47] D. Fabbri, P. Fre', L. Gualtieri, C. Reina, A. Tomasiello, A. Zaffaroni and A. Zampa, Nucl. Phys. B **577**, 547 (2000) [arXiv:hep-th/9907219].
- [48] D. Martelli and J. Sparks, JHEP **0811**, 016 (2008) [arXiv:0808.0904 [hep-th]].
- [49] C. P. Boyer and K. Galicki, Surveys Diff. Geom. **7**, 123 (1999) [arXiv:hep-th/9810250].
- [50] D. Berenstein and D. Trancanelli, Phys. Rev. D **78**, 106009 (2008) [arXiv:0808.2503 [hep-th]].
- [51] K. Hosomichi, K. M. Lee, S. Lee, S. Lee, J. Park and P. Yi, JHEP **0811**, 058 (2008) [arXiv:0809.1771 [hep-th]].
- [52] I. Klebanov, T. Klose and A. Murugan, arXiv:0809.3773 [hep-th].
- [53] C. S. Park, arXiv:0810.1075 [hep-th].
- [54] O. Aharony, O. Bergman and D. L. Jafferis, JHEP **0811**, 043 (2008) [arXiv:0807.4924 [hep-th]].
- [55] Y. Imamura and K. Kimura, Prog. Theor. Phys. **120** (2008) 509, arXiv:0806.3727 [hep-th].
- [56] M. Benna, I. Klebanov, T. Klose and M. Smedback, arXiv:0806.1519 [hep-th].
- [57] S. Terashima and F. Yagi, arXiv:0807.0368 [hep-th].
- [58] Y. Imamura, arXiv:0902.4173 [hep-th].
- [59] A. Giveon and D. Kutasov, Nucl. Phys. B **812**, 1 (2009) [arXiv:0808.0360 [hep-th]].
- [60] V. Niarchos, JHEP **0811**, 001 (2008) [arXiv:0808.2771 [hep-th]].
- [61] A. Hanany and E. Witten, Nucl. Phys. B **492**, 152 (1997) [arXiv:hep-th/9611230].
- [62] S. Corley, A. Jevicki and S. Ramgoolam, Adv. Theor. Math. Phys. **5**, 809 (2002) [arXiv:hep-th/0111222].
- [63] D. Berenstein, JHEP **0407**, 018 (2004) [arXiv:hep-th/0403110].
- [64] J. McGreevy, L. Susskind and N. Toumbas, JHEP **0006**, 008 (2000) [arXiv:hep-th/0003075].

- [65] A. Hashimoto, S. Hirano and N. Itzhaki, JHEP **0008**, 051 (2000) [arXiv:hep-th/0008016].
- [66] H. Lin, O. Lunin and J. M. Maldacena, JHEP **0410**, 025 (2004) [arXiv:hep-th/0409174].
- [67] S. Lee, S. Lee and J. Park, JHEP **0705**, 004 (2007) [arXiv:hep-th/0702120].
- [68] S. Lee, Phys. Rev. D **75**, 101901 (2007) [arXiv:hep-th/0610204].
- [69] S. Kim, S. Lee, S. Lee and J. Park, Nucl. Phys. B **797**, 340 (2008) [arXiv:0705.3540 [hep-th]].